CENTRAL VALUES OF $L$-FUNCTIONS AND HALF-INTEGRAL WEIGHT FORMS

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Abstract. We prove a relation between the Fourier coefficients of certain Hilbert modular forms of half-integral weight and central values of the corresponding Rankin $L$-functions. The approach is geometric and generalizes that of Gross and Hatcher.

1. Introduction

Let $F$ be a totally real field of degree $d$. For the purpose of this paper the class number $h(F)$ of $F$ is always assumed to be odd. Let $f$ be a holomorphic Hilbert newform over $F$ of weight $2k = (2k_1, 2k_2, \cdots, 2k_d)$ ($k_i \geq 1$) and level $N = p^e$ for a prime ideal $p$ of $F$ and $e \geq 1$. Assume further that if $d$ is odd then so is $e$. Let $B$ be the quaternion algebra over $F$ such that:

(1) If $d$ is even, then $B$ is only ramified at all infinite places,
(2) If $d$ and $e$ are both odd, then $B$ is only ramified at infinite places and $p$.

For $D \gg 0$ in $F$, let $\chi_D$ be the quadratic character of the idele class group $\mathbb{A}_F/F^\times$ associated to the quadratic extension $K_D = F(\sqrt{-D})$. Suppose the conductor of $\chi_D$ is relatively prime to $p$. Write $L(s, f_D) = L(s, f) L(s, f \otimes \chi_D)$ for the (complete) $L$-function of $f$ over $K_D$. It is well-known that the sign of the functional equation of $L(s, f_D)$ is given by $(-1)^d \chi_D(p^e)$.

Let us give a short account of the paper. In Section 2, among other things, we introduce a Gross curve $X$ and a vector bundle $V$ associated to the quaternion algebra $B$. In Section 3 we construct a Hilbert modular form $g$ of half-integral weight with coefficients in $\text{Pic}(V)$. Let $g_f$ be the $f$-isotypical component of $g$, and denote the $D$-th Fourier coefficient of $g_f$ by $m_D$. Then Section 4 Theorem 2 gives the main result of this paper, that is under certain condition on the odd fundamental discriminants $-D$, the central value of $L(s, f_D)$ is given by

$$L(1/2, f_D) = C_D \frac{(f, f)}{\langle \nu_f, \nu_f \rangle} |m_D|^2$$

where $C_D$ is an explicit positive constant and $\langle \nu_f, \nu_f \rangle$ is some height pairing, see Section 4 for the precise statement.

Formula (1.1) can be viewed as a geometric generalization of the well-known result due to Waldspurger [12] (for $F = \mathbb{Q}$). A more general but probably less explicit generalization can be found in Shimura [10]. When $F = \mathbb{Q}$, similar results have been obtained by Gross [1] (weight 2) and Hatcher [5] (higher weight). Our approach and result are extensions of theirs.

Now we make a few comments on the assumptions stated above. The oddness of the class number $h(F)$ of $F$ is used to guarantee the passage from the original central value formula, which is stated on $B^\times/F^\times$ in [13], to the current setting on $B^\times$. This assumption may be dropped if
one can prove a central value formula directly on $B^\times$. The assumption on the level of $f$ is used to eliminate the dependence of the central value formula on the choice of the $\text{Pic}(\mathcal{O}_{K_D})$ orbit of special points. This assumption can be lifted if either a central value formula involving all special points is obtained, or if the Fourier coefficients of automorphic forms of more general type, such as Jacobi forms or vector-valued forms, are considered (see [3] for a similar situation). The assumption on the parity of $d$ and $e$ is to ensure that $f$ comes from a form on $B^\times$ under the Jacquet-Langlands correspondence. It might be more natural to start with a form on $B^\times$ directly.

2. Gross curves and vector bundles

Let $B$ be the quaternion algebra over $F$ defined as in Section 1 (1) or (2). Let $R$ be a fixed order of discriminant $p^e$ over $\mathcal{O}_F$ in $B$, whose construction can be found for example in [2, Section 3] or [13, Section 3.2]. Notice that $R$ is not unique even up-to local conjugation. Let $\hat{F} = F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $\hat{B} = B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\hat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

2.1. Gross curve $X$. Let $Y$ be the genus zero curve over $F$ associated to $B$. The points of $Y$ over any $F$-algebra are given by $Y(E) = \{ \alpha \in B \otimes_F E \mid \alpha \neq 0, \text{Tr}(\alpha) = N(\alpha) = 0 \}/E^\times$.

The Gross curve $X$ is defined by the double coset

$$X = B^\times \setminus Y \times \hat{B}^\times / \hat{R}^\times .$$

Let $\hat{B}^\times = \bigcup_{i=1}^n B^\times g_i \hat{R}^\times$ be the double coset decomposition, then there is an isomorphism

$$X \cong \prod_{i=1}^n \Gamma_i \backslash Y = \prod_{i=1}^n X_i$$

where each $\Gamma_i = (B^\times \cap g_i \hat{R}^\times g_i^{-1})/\mathcal{O}_F^\times$ is a finite group with order denoted by $w_i$.

Suppose $D \gg 0$ is such that $K_D = F(\sqrt{-D})$ can be embedded into $B$. Then, there is a canonical identification $Y(K_D) \cong \text{Hom}_F(K_p, B)$ as follows: for any $f \in \text{Hom}_F(K_D, B)$, let $y$ be the image of the unique $K_D$-line on the quadric $\{ \alpha \in B \otimes_F K_D \mid \alpha \neq 0, \text{Tr}(\alpha) = N(\alpha) = 0 \}$ on which conjugation by $f(K_D^\times)$ acts by multiplication by the character $k \mapsto k/\bar{k}$. Notice that $f(K_D^\times)$ has two fixed points on $Y(K_D)$, one of which is $y$ and the other one is $\bar{y}$, the image of $y$ under the complex conjugate of $\text{Gal}(K_D/F)$. Special points on $X$ over $K_D$ are the images of $Y(K_D) \times \hat{B}^\times / \hat{R}^\times$ in $X(K_D)$.

Now suppose further that $-D \in F$ is a fundamental discriminant (Definition 3.2) and $p \nmid D$. A special point $x = (y, g) \in X(K_D)$ is said to have discriminant $-D$ if $f(K_D) \cap g \hat{R}g^{-1} = f(\mathcal{O}_{K_D})$, where $f : K \to B$ is the embedding corresponding to $y$. If the component $g$ of $x$ is congruent to $g_i$ in $B^\times \setminus \hat{R}^\times / \hat{R}^\times$, then the special point $x$ lies on the component $X_i$. Let $h_i(D)$ be the number of embeddings of $\mathcal{O}_{K_D}$ into $R_i$, modulo conjugation by $R_i^\times$. Then there are exactly $h_i(D)$ special points of discriminant $-D$ on the component $X_i$. There is a free action of the group $\text{Pic}(\mathcal{O}_{K_D}) \cong K_D^\times/B_D^\times/\hat{Q}^\times_{K_D}$ on the set of special points of discriminant $-D$ as follows. Let $x = (y, g)$ be a special point of discriminant $-D$ and let $a \in \hat{K}^\times_D$. Let $\hat{f} : \hat{K}^\times_D \to \hat{B}^\times$ be the homomorphism induced by the embedding $f : K_D \to B$ that corresponds to $y$. The action of $a$ on $x$ is then given by

$$x_a = (y, \hat{f}(a)g).$$
This action is well-defined and defines a free action of \( \text{Pic}(O_{K_D}) \) on the set of special points of discriminant \(-D\), see [1, p. 133]. The argument of [1, p. 133] shows that the orbit set classifies embeddings of \( O_{K_D, p} \) into \( R_p \) modulo conjugation by \( R_p^\times \). The later set is known [6, Theorem 5.12] or [9, Theorem 1.5.2] to have cardinality 2. On the other hand, the complex conjugate of \( \text{Gal}(K_D/F) \) on \( Y(K_D) \) induces an action on special points of discriminant \(-D\). Thus the product group \( \text{Gal}(K_D/F) \times \text{Pic}(O_{K_D}) \) acts simply transitively on the set of all special points of discriminant \(-D\) on \( X \) by counting the number of points.

The group \( \text{Pic}(X) \) of line bundles (or linearly equivalent divisor classes) on \( X \) is isomorphic to \( \mathbb{Z}^n \), and is generated by the classes \( e_i \) of degree 1 on each component \( X_i \). We define two divisor classes associated to a fundamental discriminant \(-D\) (and \( p \mid D \)). Let \( u(D) = |O_{K_D}^\times/O_F^\times| \). The first one is given by

\[
(2.4) \quad c_D = \frac{1}{2u(D)} \sum_{\text{disc}(x) = -D} \langle x \rangle,
\]

where the sum is over all special points of discriminant \(-D\). The second one is given by the \( \text{Pic}(O_{K_D}) \) orbit

\[
(2.5) \quad y = \frac{1}{u(D)} \sum_{a \in \text{Pic}(O_{K_D})} \langle x_a \rangle \in \text{Pic}(X),
\]

where \( x \in X \) is any fixed special point of discriminant \(-D\). Since \( \pi \) and \( x \) lie on the same component of \( X \), we obtain the following equality as divisor classes

\[
(2.6) \quad c_D = y = \frac{1}{2u(D)} \sum_{i=1}^{n} h_i(D)e_i \in \text{Pic}(X).
\]

If \( a = \sum_{i=1}^{n} a_ie_i \) and \( b = \sum_{i=1}^{n} b_ie_i \) are two divisor classes in \( \text{Pic}(X) \), the height pairing between them is given by

\[
(2.7) \quad \langle a, b \rangle = \sum_{i=1}^{n} w_i a_i b_i.
\]

### 2.2. Vector bundle \( V \) over \( X \).

In the following let \( v = 1, \cdots, d \) be the subscripts corresponding to infinite places of \( F \). Each local component \( B_v = B \otimes_F F_v \) with the standard Hamiltonian quaternion algebra over \( \mathbb{R} \).

Let \( W = \mathbb{C}x \oplus \mathbb{C}y \) be the standard two dimensional representation of \( SU(2) \) with an inner product given by \( [x, x] = [y, y] = 1 \) and \( [x, y] = 0 \). For each infinite place \( v = 1, \cdots, d \), the space \( \text{Sym}^{2k_v-2}(W) \) is also an inner product space with basis \( \{x^{2k_v-2}, x^{2k_v-1}y, \cdots, y^{2k_v-2}\} \). The induced inner product on \( \text{Sym}^{2k_v-2}(W) \) is such that \( [x^i j^j, x^i' y^j'] = i! j! \), see [4, Section 3] for more details. Notice that \( \text{Sym}^{2k_v-2}(W) \) is also a representation of \( SO(3) \), thus can be regarded as a representation of \( B_v^\times \). Let \( U_v \) be the vector space of trace-free elements of \( B_v \). Then \( B_v^\times \) has a natural representation on \( U_v \) by conjugation. An inner product on \( U_v \) is given by

\[
[u_1, u_2] = \frac{1}{2} \text{Tr}(u_1 u_2^\dagger). \quad \text{The space } \text{Sym}^{k_v-1}(U_v), \text{ as an inner product space, has an orthogonal decomposition } [4, \text{Section 3}]
\]

\[
(2.8) \quad \text{Sym}^{k_v-1}(U_v) = \text{Sym}^{2k_v-2}(W) \oplus M_v.
\]
Let \( W_{2k-1} = \prod_v \text{Sym}^{2k_v-2}(W) \). The space \( W_{2k-1} \) has an action of \( B^\times \) through the diagonal embedding \( B^\times \to (B \otimes \mathbb{R})^\times = \prod_v B_v^\times \). The vector bundle \( V \) is then defined by

\[
V = \left( B^\times \setminus Y \right) \times W_{2k-1} \times \hat{B}^\times / \hat{R}^\times.
\]

If we write out the double coset quotient then

\[
\prod_{i=1}^n \Gamma_i \setminus (Y \times W_{2k-1}) = \prod_{i=1}^n V_i.
\]

For each \( i \), let \( W^{\Gamma_i}_{2k-1} \) be the elements invariant under the action of \( \Gamma_i \), then \( \text{Pic}(V) \) is defined by

\[
\text{Pic}(V) = \bigoplus_{i=1}^n \mathbb{C} \otimes W^{\Gamma_i}_{2k-1}.
\]

Notice that \( \text{Pic}(V) \) is already a vector space over \( \mathbb{C} \). Let \( \nu = (y, g, w) \in V \), define the class of \( \nu \) to be

\[

(\nu) = \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} e_i \otimes \gamma(w) \in \text{Pic}(V).
\]

The height pairing on \( V \times V \) is defined as follows. Let \( \nu_1 = (y_1, g_1, w_1) \) and \( \nu_2 = (y_2, g_j, w_2) \), then

\[
\langle \nu_1, \nu_2 \rangle = \begin{cases} 0, & i \neq j \\ \sum_{\gamma \in \Gamma_i} [w_1, \gamma(w_2)] w_{2k-1}, & i = j \end{cases}
\]

Formula (2.13) induces a height pairing on \( \text{Pic}(V) \times \text{Pic}(V) \), which coincides with (2.7) if \( k_v = 2 \) for all \( v \).

Let \( D \gg 0 \) be such that there is an embedding \( f : K_D \to B \). At each infinite place \( v \), let \( u_v^D = \sqrt{-D_v} \in U_v \), and let \( w_v^D \) be the component of \( (u_v^D)^{k_v-1} \) in \( \text{Sym}^{2k_v-2}(W) \) through (2.8). Write \( w_v^D = \prod_v w_v^D \in W_{2k-1} \). A straight computation [4, p. 556] gives

\[
[w_0^D, w_0^D] = \prod_v \frac{2^{2k_v-2} D_v^{k_v-1}(k_v-1)!^3}{(2k_v-2)!}.
\]

Suppose further that \( -D \) is a fundamental discriminant with \( p \nmid D \). A special point on \( V \) of discriminant \( -D \) is a point of the form \( \nu = (y, g, w_v^D) \), where \( x = (y, g) \) is a special point on \( X \) of discriminant \( -D \). The action of \( \text{Pic}(\mathcal{O}_{K_D}) \) on special points of \( X \) extends naturally to special points of \( V \) through \( \nu_a = (x_a, w_0^D) \), see (2.3). Similar to (2.6) define the special cycle of \( V \) by

\[
\nu_D = \frac{1}{u(D)} \sum_{a \in \text{Pic}(\mathcal{O}_{K_D})} (\nu_a) = \frac{1}{2u(D)} \sum_{i=1}^n h_i(D)(e_i \otimes w_0^D) \in \text{Pic}(V).
\]

2.3. Curve \( X' \) and vector bundle \( V' \). To utilize the central value formula of [13] we need to define a new curve \( X' \) and a vector bundle \( V' \) on it. The construction is parallel to that in Section 2.2. After the construction the relationship between them is studied.

The curve \( X' \) is defined by

\[
X' = B^\times \setminus Y \times \hat{B}^\times / \hat{R}^\times.
\]
If we write \( \hat{B}^\times = \cup_{j=1}^m B^\times g_j^\times \tilde{R}^\times \) for the double coset decomposition, then

\[
(2.17) \quad X' \cong \prod_{j=1}^m \Gamma'_j \backslash Y,
\]

where \( \Gamma'_j \cong (B^\times \cap g_j^\times \tilde{R}^\times g_j^{-1})/F^\times \).

Thus the group of line bundles \( \text{Pic}(X') \) of \( X' \) is isomorphic to \( \mathbb{Z}^m \). If \( a' = \sum_{j=1}^m a_j e'_j \) and \( b' = \sum_{j=1}^m b_j e'_j \), the height pairing between them is given by

\[
(2.18) \quad \langle a', b' \rangle' = \sum_{j=1}^m w'_j a_j b_j.
\]

**Assumption:** From now on we will assume the class number \( h(F) \) of \( F \) is odd.

**Lemma 2.1.** The (multiplication) action of \( \text{Pic}(\mathcal{O}_F) = F^\times \backslash \tilde{F}^\times / \tilde{\mathcal{O}}_F^\times \) on \( S = B^\times \backslash \hat{B}^\times / \hat{R}^\times \) is free. The quotient space is given by

\[
B^\times \backslash \tilde{B}^\times / \tilde{R}^\times \hat{R}^\times.
\]

**Proof.** The second claim is obvious, and we only need to prove the first one. Let \( t \in \tilde{F}^\times \) be a stabilizer of \( g \in \hat{B}^\times \) such that \( gt = agr \) with \( a \in B^\times \) and \( r \in \hat{R}^\times \). Then \( N(g)N(t) = N(a)N(g)N(r) \), which implies that \( t^2 = N(t) = N(a)N(r) \in F^\times \tilde{\mathcal{O}}_F^\times \). Since \( h(F) \) is assumed to be odd, we conclude that \( t \) itself has to define a trivial class in \( \text{Pic}(\mathcal{O}_F) \). \( \square \)

By Lemma 2.1 we can and will take \( g'_j = g_j \) for \( j = 1, \ldots, m \). Also, \( n = mh(F) \).

**Remark 2.1.** The same argument shows that the natural homomorphism \( \text{Pic}(\mathcal{O}_F) \rightarrow \text{Pic}(\mathcal{O}_K) \) is injective for any quadratic extension \( K/F \).

**Lemma 2.2.** For each \( j = 1, \ldots, m \), there is a natural isomorphism from \( \Gamma_j \) to \( \Gamma'_j \). More precisely, the natural homomorphism

\[
(2.19) \quad \left( B^\times \cap g_j \tilde{R}^\times g_j^{-1} \right) / \mathcal{O}_F^\times \cong \left( B^\times \cap g_j \tilde{F}^\times \tilde{R}^\times g_j^{-1} \right) / F^\times.
\]

is an isomorphism. Thus \( w'_j = w_j \) for \( j = 1, \ldots, m \).

**Proof.** The homomorphism in (2.19) is induced by the identity map. To see it is surjective: suppose \( b \in B^\times \cap g_j \tilde{F}^\times \tilde{R}^\times g_j^{-1} \). Then we need to find \( b_1 \in B^\times \cap g_j \tilde{R}^\times g_j^{-1} \) such that \( b_1 = bf \) for some \( f \in F^\times \). Since \( h(F) \) is odd, and taking the norms of \( b = g_j tr g_j^{-1} \in B^\times \) we get \( N(b) = t^2 N(r) \in F^\times \), which implies that \( t \in F^\times \tilde{\mathcal{O}}_F^\times \), so \( b_1 = bf \) for some \( f \in F \). For injectivity, suppose \( b_1, b_2 \in B^\times \cap g_j \tilde{R}^\times g_j^{-1} \) such that \( b_1 = fb_2 \). Since \( b_1 b_2^{-1} \in g_j \tilde{R}^\times g_j^{-1} \), we get that \( f \in F^\times \cap g_j \tilde{R}^\times g_j^{-1} = \mathcal{O}^\times \). \( \square \)

**Corollary 2.3.** The natural projection

\[
\pi: X = B^\times \backslash Y \times \tilde{B}^\times / \hat{R}^\times \rightarrow X' = B^\times \backslash Y \times \tilde{B}^\times / \tilde{R}^\times \hat{R}^\times
\]

is an étale covering of equal degree \( h(F) \).
Corollary 2.4. Let \( c_D = \pi(c_D) \in \text{Pic}(X') \), then
\[
\pi^*(c_D) = c_D.
\]
Thus \( \langle c_D, c_D \rangle = \deg(\pi)\langle c_D', c_D' \rangle = h(F)\langle c_D', c_D' \rangle \) by the projection formula.

Notice that the centers of \((B \otimes \mathbb{R})^x\) and \(B_v^x\) act trivially on \( W_{2k-1} \) and \( U_v \) (Section 2.2) respectively. Thus the vector bundle \( V \) descends to a vector bundle on \( X' \), and is denoted by \( V' \). The group \( \text{Pic}(V') \) can also be defined similarly. \( \text{Pic}(V) \) is equipped with a height pairing, denoted again by \( \langle \cdot, \cdot \rangle \). The projection of a special point \( \nu \) of fundamental discriminant \( -D \) on \( V \) defines a special point \( \nu' \) on \( V' \). Hence the special cycle \( \nu_D \) (2.15) descends to a special cycle on \( V \)
\[
\nu'_D = \sum_{a \in \text{Pic}(\mathcal{O}_{K_D})/\text{Pic}(\mathcal{O}_F)} (\nu'_a) = \frac{1}{2u(D)} \sum_{j=1}^m h_j(D)(e'_j \otimes w_0^D) \in \text{Pic}(V').
\]

Similar to Corollary 2.4, we have the following.

Proposition 2.5. Let \( \nu_D \) and \( \nu'_D \) be defined in (2.15) and (2.21) respectively. Then
\[
\langle \nu_D, \nu_D \rangle = h(F)\langle \nu'_D, \nu'_D \rangle.
\]

3. Hilbert modular forms of half-integral weight

Let \( B^0 \) be the subspace of trace-free elements of \( B \), and let \( U = \prod_v B^0 \otimes_v \mathbb{R} \). For each infinite place \( v \), let \( \mu^1_v, \mu^2_v, \mu^3_v \) be a fixed basis of \( B^0 \otimes_v \mathbb{R} \). A homogeneous polynomial \( P = \prod_v P_v \) of degree \( k-1 \) on \( U \) is called spherical harmonic if for each \( v \)
\[
\sum_{i=1}^3 \frac{\partial^2}{\partial x_{i,v}^2} P_v = 0.
\]

Lemma 3.1. For each \( \nu \in \text{Pic}(V) \), there is spherical harmonic polynomial \( P_{\nu}^\nu = \prod_v P_{i,v}^\nu \) of degree \( k-1 \) on \( U \), such that for every \( D \gg 0 \), one has
\[
\langle \nu, (e_i \otimes w_0^D) \rangle = \prod_v P_{i,v}^\nu(\sqrt{-D_v}) = \prod_v P_{i,v}^\nu(x_{1,v}, x_{2,v}, x_{3,v}),
\]
where \( \sqrt{-D_v} = x_{1,v} \mu^1_v + x_{2,v} \mu^2_v + x_{3,v} \mu^3_v \).

Proof. Here \( w_0^D \) is defined in Section 2.2. See [8, Proposition 2.10] or [4, Lemma1] for a proof. \( \square \)

Let \( R_i, i = 1, \cdots, n \) be the order associated to \( g_i \in B^x \backslash \tilde{B}^x/\tilde{R}^x \), that is \( R_i = B \cap g_i \tilde{R} g_i^{-1} \). Let
\[
S_i = \mathcal{O}_F + 2R_i
\]
be a sub-order of \( R_i \). Write \( S_i^0 \) for the subset of trace-free elements of \( S_i \). For each \( D \gg 0 \) in \( \mathcal{O}_F \), define \( A_i(D) = \{ b \in S_i^0 \mid N(b) = -b^2 = D \} \) and \( a_i(D) = |A_i(D)| \).

Definition 3.2. Let \( D \gg 0 \) be in \( \mathcal{O}_F \) and \( K_D = F(\sqrt{-D}) \). Then \( -D \) is called a fundamental discriminant if \( \mathcal{O}_{K_D} \) has relative discriminant \( (D) \) over \( \mathcal{O}_F \), and
\[
\mathcal{O}_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F
\]
for some \( a \in \mathcal{O}_F \).
Proposition 3.3. Suppose the relative discriminant of $K_D$ over $F$ is $(D)$, and suppose $K_D$ splits over all primes of $F$ dividing 2. Then $-D$ is a fundamental discriminant.

Proof. First, notice that $x = \frac{a + \sqrt{-D}}{2}$ for $a \in \mathcal{O}_F$ is in $\mathcal{O}_{K_D}$ if and only if $4|(a^2 + D)$.

Let $(2) = p_1^{e_1} \cdots p_g^{e_g}$ be the prime decomposition of $(2)$ in $\mathcal{O}_F$. Since each $p_i$ splits in $K_D$, the equation $y_i^2 = -D$ has solutions in $\mathcal{O}_{F_{p_i}}$, which modulo $p_i^{2e_i}$ implies that $y_i^2 \equiv -D \pmod{p_i^{2e_i}}$ has solutions for all $i$. Now the Chinese Remainder Theorem tells us that there is $a \in \mathcal{O}_F$ such $a \equiv y_i$ for all $i$. Such $a$ satisfies the required property $a^2 \equiv -D \pmod{4}$. Hence $x = \frac{a + \sqrt{-D}}{2} \in \mathcal{O}_{K_D}$.

Let $\mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$. As $\left(\frac{a + \sqrt{-D}}{2}\right)^2 = a^2 - D + a^2 - D + a^2 + \sqrt{-D} \in \mathcal{O}$, so $\mathcal{O}$ is an order in $K$. Moreover, the discriminant of $\mathcal{O}$ is $(D)$ which is the same as that of $\mathcal{O}_K$. Therefore $\mathcal{O}_K = \mathcal{O} = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$ and $-D$ is a fundamental discriminant.

Proposition 3.3 implies that there are plenty of fundamental discriminants. For instance, let $D$ be square-free such that $(D, d_F) = 1$ and $-D \equiv 1 \pmod{8}$, then $-D$ is a fundamental discriminant.

Proposition 3.4. Suppose $-D$ is a fundamental discriminant. Then, for each $i = 1, \ldots, n$

\[
\frac{a_i(D)}{w_i} = h_i(D) \frac{u(D)}{u(D)},
\]

where $h_i(D)$ is the number of (optimal) embeddings of $\mathcal{O}_K$ into $R_i$, modulo conjugation by $R_i^\times$.

Proof. First let $f : \mathcal{O}_K \rightarrow R_i$ be an embedding. Write $\mathcal{O}_K = \mathcal{O}_F + \frac{a + \sqrt{-D}}{2} \mathcal{O}_F$. The element $b = f(\sqrt{-D}) \in R_i$ then satisfies $\text{Tr}(b) = 0$ and $\mathcal{N}(b) = -D$. Since $\frac{a + b}{2} = f\left(\frac{a + \sqrt{-D}}{2}\right) \in R_i$, we have $b \in A_i(D)$.

Conversely let $b \in A_i(D)$. Thus $a' + b \in 2R_i$ for some $a' \in \mathcal{O}_F$. In particular, $\frac{a' + b}{2}$ is integral over $\mathcal{O}_F$, which implies that $\frac{a' + \sqrt{-D}}{2} \in K_D$ is integral over $K_D$, that is $\frac{a' + \sqrt{-D}}{2} \in \mathcal{O}_K$. By comparing discriminants we get

\[
\mathcal{O}_K = \mathcal{O}_F + \frac{a' + \sqrt{-D}}{2} \mathcal{O}_F.
\]

Thus we obtain an embedding $f : \mathcal{O}_K \rightarrow R_i$ by letting $f\left(\frac{a + \sqrt{-D}}{2}\right) = \frac{a' + b}{2}$.

The group $\Gamma_i = R_i^\times / \mathcal{O}_F^\times$ acts on $A_i(D)$ and the set of embeddings by conjugation, hence we have proved

\[
|A_i(D)/\Gamma_i| = h_i(D).
\]

Now (3.2) follows form (3.3) because the order of the stabilizer of an element $b \in A_i(D)$ under the action of $\Gamma_i$ is equal to $u(D)$.

Proposition 3.5. The series

\[
g = \sum_{D > 0, \text{or } D = 0} \left( \sum_{i=1}^{n} \frac{a_i(D)}{2w_i} (e_i \otimes w_0^D) \right) q^D
\]

defines a Hilbert modular form of weight $k + \frac{1}{2}$ with coefficients in Pic($V$). Here $z = (z_1, z_2, \ldots, z_d) \in \mathbb{H}^d$, and $q^D = \exp(2\pi i \text{Tr}(Dz))$. Moreover, if $-D$ with $p \mid D$ is a fundamental discriminant, then the $D$-th Fourier coefficient of $g$ equals $v_D$.\]
Proof. By Lemma 3.1, the \(i\)-th series in (3.4) is a theta series with spherical harmonic coefficients attached to the quadratic space \(S_0^i\). The modularity and weight of such series is well-known when \(F = \mathbb{Q}\), see for instance [7]. For a general field \(F\), the proof can be found in [11, Section 5].

If \(-D\) is a fundamental discriminant, the equality between the \(D\)-th Fourier coefficient of \(g\) and \(\nu_D\) follows by comparing (2.15) and (3.2).

The term for \(D = 0\) in (3.4) is nonzero only if \(k_v = 2\) for all \(v\).

4. Central values and Fourier coefficients of half-integral weight forms

In this section we first recall a formula for the central value \(L(1/2, f_D)\) which is expressed in terms of the height pairing between special cycles of \(\text{Pic}(V)\). Then we relate these central values to the Fourier coefficients of the Hilbert modular form of half-integral weight constructed in Proposition 3.5.

From now on let \(-D\) with \(p \nmid D\) be a fundamental discriminant, and assume there exists a special point of discriminant \(-D\) on \(X\).

**Theorem 1.** The central value of \(L(s, f_D)\) is given by

\[
L(1/2, f_D) = C_D (f, f) \langle \nu_D, f \rangle,
\]

where \(C\) is given by

\[
C_D = C_1 h(F) d_F^{3/2} d_{K_D}^{-1/2} \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}}
\]

with \(C_1\) defined in [13, (1.4)], \((f, f)\) is the Petersson inner product, and \(\nu_D, f\) is the \(f\)-isotypical component of \(\nu_D\).

**Proof.** By [13, Theorem 1.2]

\[
L(1/2, f_D) = C_1 h(F) d_F^{3/2} d_{K_D}^{-1/2} (f, f) \langle \nu_D, f \rangle_G
\]

where \(C_1\) is the rational number given by [13, (1.4)], and \(\langle \cdot, \cdot \rangle_G\) is the geometric pairing defined in [13, Section 3.1] through a multiplicity function \(M_\infty = \prod_v M_v\), which is defined as follows. Choose a decomposition \(B = K_D + K_D J\), and let \(\xi(a + bJ) = \frac{N(b)}{N(a + bJ)}\), then

\[
M_v(\gamma_v) = \frac{2^{2k_v-1}(k_v - 1)!}{(2k_v - 2)!} P_{k_v-1}(1 - 2\xi(\gamma_v)),
\]

By [4, (7.6)] one has

\[
[w_0^D, \gamma(w_0^D)] = \left( \prod_v P_{k_v-1}(1 - 2\xi(\gamma_v)) \right) [w_0^D, w_0^D]
\]

\[
= \prod_v P_{k_v-1}(1 - 2\xi(\gamma_v)) \cdot \prod_v \frac{2^{2k_v-1}(k_v - 1)!}{(2k_v - 2)!}
\]

\[
= \left( \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) M_\infty(\gamma_v).
\]
Comparing the height pairing (2.13) and the geometric pairing in [13, Section 3.1] we obtain the following relation between them

\[ \langle \nu_{D,f}', \nu_{D,f}' \rangle' = \left( \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) \langle \nu_{D,f}', \nu_{D,f}' \rangle_G \]

By Proposition 2.5

\[ \langle \nu_{D,f}, \nu_{D,f} \rangle = h(F) \left( \prod_v \frac{(k_v - 1)!}{2D_v^{1-k_v}} \right) \langle \nu_{D,f}', \nu_{D,f}' \rangle_G, \]

which combined with (4.3) completes the proof of (4.1). \[ \square \]

Remark 4.1. (1) Strictly speaking, the special cycle in [13] is defined over a curve associated to the group \( G = B^\times / F^\times \), but there is an isomorphism

\[ B^\times \backslash \hat{B}^\times / \hat{F}^\times \cong G(F) \backslash G(\mathbb{A}_f) / \Delta, \]

with \( \Delta = \hat{F}^\times / \hat{O}_F^\times \). This isomorphism is induced by \( \hat{B}^\times / \hat{F}^\times \cong G(\mathbb{A}_f) \), and \( \hat{F}^\times \cap \hat{\mathbb{O}}_F^\times = \hat{O}_F^\times \).

(2) The existence of a special point of discriminant \(-D\) implies that the functional equation of \( L(s, f_D) \) has sign +1 (due to the assumption on the parity of \( d \) and \( e \)). Hence the result of [13] can be applied.

(3) There are naturally defined Hecke operators acting on all the curves and line bundles defined in Section 2. By the Jacquet-Langlands correspondence \( f \) determines an eigenform on the group \( B^\times \backslash B^\times(\mathbb{A}_F) \) which has the same Hecke eigenvalues as \( f \). Hence it is legitimate to speak of \( f \)-isotypical components on \( \text{Pic}(V) \).

Let \( \nu_f \in \text{Pic}(V) \otimes \mathbb{C} \) be a nonzero element in the \( f \)-isotypical component. By strong multiplicity one theorem such \( \nu_f \) is unique up-to a scalar multiple. Let

\[ g(\nu_f) = \sum_D m_D q^D = \langle g, \nu_f \rangle. \]

Now we state the main result of this paper.

Theorem 2. Let \(-D\) with \( p \nmid D \) be an odd fundamental discriminant such that there is a special point on \( X \) of discriminant \(-D\). Then

\[ L(1/2, f_D) = C_D \frac{\langle f, f \rangle_{\nu_f, \nu_f}}{\langle \nu_f, \nu_f \rangle} |m_D|^2, \]

where \( C_D \) is given by (4.2).

Proof. By Proposition 3.5 and (4.8)

\[ m_D = \langle \nu_D, \nu_f \rangle = \langle \nu_{D,f}, \nu_f \rangle. \]

Thus

\[ \nu_{D,f} = \frac{m_D}{\langle \nu_f, \nu_f \rangle} \nu_f \in \text{Pic}(V). \]

So

\[ \langle \nu_{D,f}, \nu_{D,f} \rangle = \frac{|m_D|^2}{\langle \nu_f, \nu_f \rangle}. \]

Now (4.9) follows immediately from (4.1). \[ \square \]
The form $g(\nu f)$ is a Niwa-Shintani lifting of $f$, hence (4.9) can be regarded as a geometric generalization of the result of [12].

REFERENCES


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